Power-law and exponential segregation in two-dimensional silos of granular mixtures

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When a binary mixture of granular materials, differing in shape or size, is poured into a quasi-two-dimensional silo, segregation of the mixture is observed. Depending on the size ratio d_2/d_1 of the species, the mixture segregates completely or partially into the pure species. To study the partial-segregation effect we propose a theoretical model based on the work of Boutreux and de Gennes [J. Phys. I **6**, 1295 (1996)] but we introduce more realistic collision functions. To compare the partial- and complete-segregation effects, we also discuss calculations for the complete-segregation model proposed by Makse [Phys. Rev. E **56**, 7008 (1997)]. Our experiments confirm the analytical solutions for both types of segregation. We find that the transition from complete segregation to partial segregation appears as the size ratio of the species is decreased below a critical value, which is found to be $d_2/d_1 \simeq 1.4$ for our system. Our experimental and analytical studies predict the regime for applicability of both partial- and complete-segregation models in terms of the size ratio of the species and the respective model parameters.

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I. INTRODUCTION

Size segregation is one of the interesting and unusual properties of granular mixtures [1–5] which is known to occur when a mixture of granular materials is exposed to external periodic perturbations such as vibrations [6–10] or rotations [11–13]. Size segregation can also occur in the absence of external perturbations. For example, when a mixture of grains of different size is poured onto a heap, the larger grains are more likely to be segregated near the bottom of the heap while the smaller grains segregate near the pouring point of the heap [14–18]. Different forms of segregation are observed when the grains differ in size and surface properties (shape, roughness, etc.). It has recently been shown that when the mixture is composed of grains differing not only in size but also in shape, another type of segregation, known as stratification, may be observed [19–23]. When the mixture is composed of large grains that are more faceted, such as cubical, and small grains that are less faceted, such as spherical, the mixture spontaneously stratifies into alternating layers of larger faceted and smaller rounded grains. A model of the stratification process is shown in Fig. 1(a).

In contrast, when the mixture is composed of large rounded grains and small more faceted grains, only segregation of the mixture results. The larger and smoother grains preferentially stop near the bottom whereas the smaller grains are more likely to be found at the top of the pile [19,21,22]. When the grains have large differences in size, complete segregation is observed in the steady state. In this case, the small grains segregate near the top of the pile whereas the large grains segregate near the bottom. Figure 1(b) shows a model of complete segregation. When the mixture of grains does not differ much in size and shape, then partial segregation is observed [21,22]. In this type of segregation, the larger and smoother grains preferentially stop near the bottom whereas the smaller grains stop at the top of the pile. The degree of the segregation is small here since grains of both types are present almost everywhere in the pile.

According to previous work by Makse *et al.* [19], the control parameter for the stratification-segregation transition is the difference between the repose angle of the pure species,

$$\delta \equiv \theta_{22} - \theta_{11}, \qquad (1)$$

where θ_{11} is the angle of repose of the small grains and θ_{22} is the angle of repose of the large grains. When the difference of the repose angles is positive—i.e., $\delta > 0$—then the stratification process occurs. On the other hand, complete segregation occurs when $\delta < 0$. In addition, when the size ratio of the species is close to 1, partial segregation of the mixture is supposed to occur, irrespective of the difference in the repose angle.

The segregation processes have been studied theoretically by many groups. Bouchaud, Cates, Ravi Prakash, and Edwards (BCRE) [24,25] developed a theoretical continuum approach to describe the case of a single species in a two-dimensional geometry. Later, de Gennes [26] applied the BCRE formalism to study the case of granular flows in a thin rotating drum. Boutreux and de Gennes (BdG) [27] treated the case of granular flows made of two species of different angles of repose by generalizing the BCRE equations and developing a minimal model that predicts a power-law behavior of the concentrations in partial segregation. Later, Makse and co-workers [28,29] used the continuum approach of BdG [27] to study segregation and stratification processes analytically. They derived steady-state solutions to the equations of motion for a two-species granular flow when the two species have a different angle of repose and performed a stability analysis predicting complete-segregation and stratification processes. Their analysis was later applied to study the segregation process in a thin rotating drum [30]. In that case, complete segregation with a small region of mixing was observed for a large size ratio of the species, while a slow power-law decay of the concentrations of the grains was predicted when the species differ slightly in size.

Most of the previous work has been analytical (with the exception of Refs. [20,22]); thus, experimental studies are needed to test the different predictions and further develop the theory. In this work we present experimental and analyti-

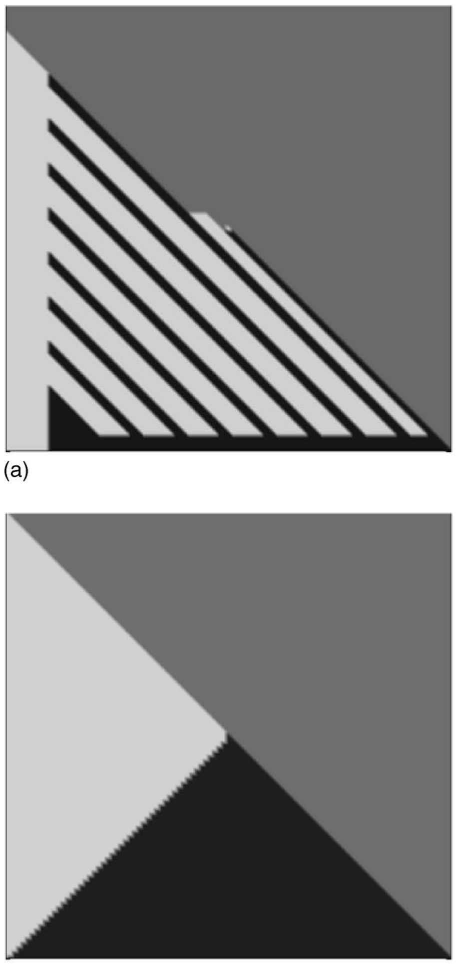

FIG. 1. Different forms of segregation are observed when a mixture of granular grains differing in shape or size is poured into a two-dimensional geometry. (a) Model of stratification for a mixture of smaller rounded grains and larger faceted grains. (b) Model of complete segregation for a mixture of smaller faceted grains and larger rounded grains.

cal studies of the partial- and complete-segregation processes in a quasi-two-dimensional pile formed by pouring grains of different size. Our paper is organized as follows. Section II deals with the analytical part of our work where we propose simplified models other than the minimal model for partial and complete segregation [27] in order to understand the concentration profiles for both cases. In this context, we first recall the basic BCRE [24,25] and BdG [27] approaches to understand the flow behavior of granular materials and then we find the steady-state solution analytically for a two-species granular flow in a quasi-two-dimensional silo for the case of partial segregation. In Sec. III, we perform experiments to study both types of segregation by varying the size ratio of the species. There, we also analyze the experimental data for the concentration profiles and compare them with the analytical solutions to determine the relevant fitting parameters. The experimental profiles for both types of segregation patterns are well fitted with our analytical solutions of the granular flow equations. The partial-segregation profiles are characterized by a power-law behavior of the concentration while complete-segregation profiles follow exponential functions. Our comparison of experiment and theory predicts the regime for applicability of partial- and complete-segregation models in terms of the size ratio of the species. We find partial segregation of the mixtures with a size ratio less than 1.4, while for large size ratio the segregation is complete.

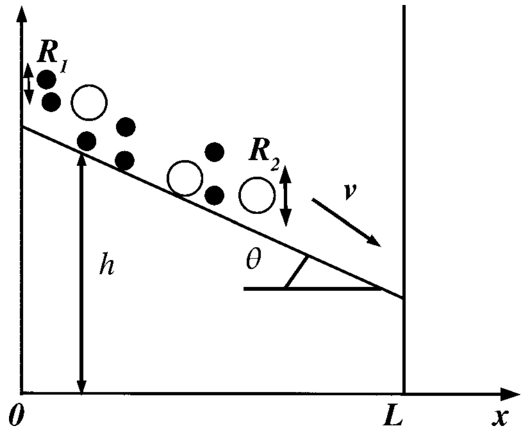

FIG. 2. Diagram showing the variables used to describe the granular flow of mixtures in a two-dimensional silo. We use the label 1 for smaller grains and label 2 for larger grains.

II. ANALYTICAL STUDY

A. Continuum theory for surface flow of granular mixtures

BCRE [24,25] described the dynamics of two-dimensional sandpile surfaces for the case of a single species. They proposed two coupled variables: the local angle of sandpile $\theta(x,t)$ to describe the static phase [alternatively the height of the sandpile $h(x,t)$, where $\theta(x,t)=-\partial h(x,t)/\partial x$], and the local thickness of the layer of the rolling grains $R(x,t)$ to describe the rolling phase (see Fig. 2). Both $\theta(x,t)$ and $R(x,t)$ are considered to be a function of time t and the longitudinal coordinate x. BCRE also proposed a set of convection-diffusion equations for the rolling grains and found solutions for the case of a single-species granular flow. Later BdG [27] extended the BCRE formalism to treat the case of a two-species granular flow. In the BdG formalism, two local *equivalent thicknesses* of the species in the rolling phase $R_i(x,t)$ were considered, with $i=1,2$, respectively, for the small and large grains (Fig. 2). In our calculations, we will use the same labels for the species—i.e., $i=1$ for the small grains and $i=2$ for the large grains. The total thickness of the rolling phase is then defined as

$$R(x,t) \equiv R_1(x,t) + R_2(x,t). \tag{2}$$

The static phase is described by the height of the pile and the volume fraction of the static grains $\phi_i(x,t)$ of type i at the surface of the pile. Thus we have

$$\phi_1(x,t) + \phi_2(x,t) = 1. \tag{3}$$

The equation of motion for the rolling species is [27]

$$\frac{\partial R_i(x,t)}{\partial t} = - v_i \frac{\partial R_i(x,t)}{\partial x} + \Gamma_i. \quad (4)$$

The first term on the right-hand side of the equation takes into account the downhill convection of grains due to gravity, where v_i is the downhill convection speed of the rolling grain i. The Γ_i term stands for the interaction between the static and the rolling grains. Furthermore, the equation of mass conservation yields

$$\phi_i(x,t)\frac{\partial h(x,t)}{\partial t} = - \Gamma_i. \quad (5)$$

Sum over i in Eq. (5), and we obtain an equation for h:

$$\frac{\partial h}{\partial t} = - \Gamma_1 - \Gamma_2. \quad (6)$$

BCRE proposed a form for Γ_i in the general case. Later, BdG simplified it for the case of a continuous flow of rolling grains. We assume the BdG formalism for the interaction term

$$\Gamma_i = a_i(\theta)\phi_i R_i - b_i(\theta)R_i + x_j(\theta)\phi_i R_j. \quad (7)$$

Thus, Γ_i is a function of the collision functions which contribute to the rate processes. These functions $a_i(\theta)$, $b_i(\theta)$, and $x_i(\theta)$ were previously defined by BdG as the following: $a_i(\theta)$ is the contribution due to an amplification process (i.e., when a static grain of type i is converted into a rolling grain due to a collision by a rolling grain of type i), $b_i(\theta)$ is the contribution due to capture of a rolling grain (i.e., when a rolling grain of type i is converted into a static grain), and $x_i(\theta)$ is the contribution due to a cross-amplification process (i.e., the amplification of a static grain of type j due to a collision by a rolling grain of type i).

The collision functions fully define the dynamics of the system. Since they are *a priori* unknown, they have to be defined following certain assumptions and approximations. BdG proposed a minimal model for the segregation where the cross amplification $x_i(\theta)$ is held constant and the dependence of the angle of repose θ on the concentration ϕ_j is neglected. In this work we release the approximations made in the minimal model and propose general collision functions. Our approach allows us to treat the cases of both partial and complete segregation in two-dimensional silos.

B. Steady-state solutions

To calculate the steady-state solution of the equation of motion for the granular flow of two species in a quasi-two-dimensional geometry, a silo of lateral size L is considered and the pouring point is assumed to be at $x=0$ (see Fig. 2). Also, the convection speeds for both types of grains are set to be equal and constant, i.e., $v_i=v$. At steady state, the rolling phase is time invariant while the static phase grows at constant speed since we are filling the silo continuously at the top. Thus, we have

$$\frac{\partial R_i}{\partial t} = 0 \quad (8)$$

and

$$\frac{\partial h}{\partial t} = \frac{vR^0}{L}, \quad (9)$$

where R^0 is defined by the following boundary conditions:

$$R_i(x=0) = R_i^0, \quad (10)$$

$$R_i(x=L) = 0, \quad (11)$$

and

$$R^0 = R_1^0 + R_2^0. \quad (12)$$

At steady state, Eq. (4) takes the form

$$0 = - v\frac{\partial R_i}{\partial x} + \Gamma_i. \quad (13)$$

From Eqs. (6) and (9), we obtain

$$\Gamma_1 + \Gamma_2 = - \frac{vR^0}{L}. \quad (14)$$

Summing over i in Eq. (13) and substituting the value of $\Gamma_1+\Gamma_2$ from Eq. (14), Eq. (13) can be rearranged as follows:

$$0 = - v\frac{\partial R}{\partial x} - \frac{vR^0}{L}. \quad (15)$$

Equation (15) is solved with the boundary conditions [Eqs. (10)–(12)], and the result is a linearly decaying profile of total rolling species with x,

$$R = \frac{R^0}{L}(L - x). \quad (16)$$

Defining $R_i^* \equiv R_i/R$ and using Eqs. (5) and (9) in Eq. (13), we obtain

$$(L - x)\frac{\partial R_i^*}{\partial x} = R_i^* - \phi_i. \quad (17)$$

In what follows, we will consider two separate cases for partial and complete segregation. First the steady-state solution for partial segregation will be developed. Later, previous results of complete segregation will be reviewed along with the ultimate goal of testing the predictions of both theories of segregation.

C. Steady-state solution for partial segregation

When the difference in size or, in general, the difference in the angle of repose of the grains is not too large (we will quantify the word *large* later), the steady-state solution yields partial or weak segregation. As mentioned earlier, a minimal model to study partial segregation was proposed by BdG, where the cross amplification $x_i(\theta)$ [Eq. (7)] was held constant and the dependence of the angle of repose θ on the concentration of grains in the pile (ϕ_j) was neglected. The minimal model was solved in steady state with these approximations and produced a surprising power-law dependence of the concentration of the species on the longitudinal coordinate, $\phi_i(x)$. However, later it was shown that the dependence of ϕ_i on the repose angles is crucial to understand

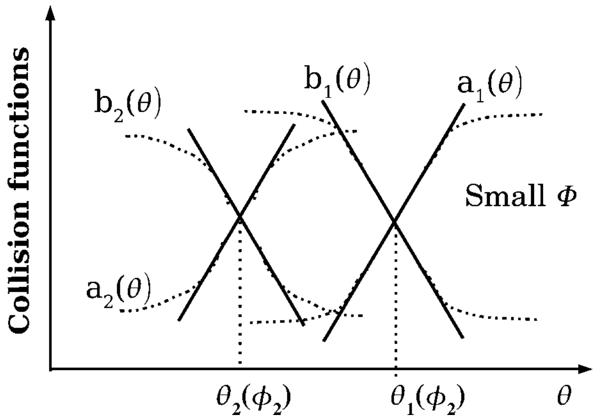

FIG. 3. Linear approximation (solid lines) of the collision functions (dashed lines) when the species do not differ much in shape and size. The approximation is plausible in the vicinity of the angle of repose only (Ref. [30]).

other segregation effects such as stratification and complete segregation [19,20,30]. In a previous work, Makse [30] released the approximations made in the minimal model and studied segregation of granular mixtures in a thin rotating drum by linearizing the collision functions in the vicinity of the angle of repose (see Fig. 3). To treat the case of partial segregation in a quasi-two-dimensional silo, we use the same definitions of the collision functions, as proposed in Ref. [30]:

$$a_i(\theta) \equiv x_i(\theta) \equiv C + \gamma[\theta(x) - \theta_i(\phi_j)],$$

$$b_i(\theta) \equiv C - \gamma[\theta(x) - \theta_i(\phi_j)]. \tag{18}$$

Here γ and C are collision rates. $\theta_i(\phi_j)$ is the generalized angle of repose of a rolling grain of type i $[i,j=1,2,(j \neq i)]$. $\theta_i(\phi_j)$ depends linearly on the composition of the surface ϕ_j [29] (see Fig. 4):

$$\theta_1(\phi_2) = m_1\phi_2 + \theta_{11},$$

$$\theta_2(\phi_2) = m_1\phi_2 + \theta_{21} = -m_1\phi_1 + \theta_{22}. \tag{19}$$

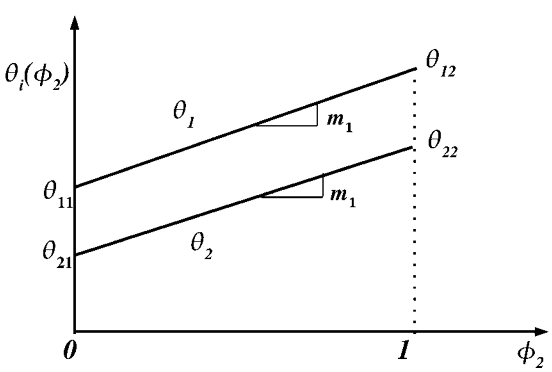

FIG. 4. Dependence of the generalized angle of repose θ_i for the two rolling species on the concentration of the surface of large grains ϕ_2 when the difference of the repose angles is positive—i.e., $\delta > 0$ (Ref. [29]).

The limiting cases are $\theta_{ij} = \theta_i(\phi_j = 1)$, and the slope is defined as $m_1 \equiv \theta_{12} - \theta_{11} = \theta_{22} - \theta_{21}$. Here, θ_{11} and θ_{22} are the angles of repose of the pure species, while θ_{12} and θ_{21} are the cross angles of repose (θ_{12}: cross angle of repose of a grain 1 on top of a grain 2 and vice versa for θ_{21}). From Fig. 4, it is clear that the difference

$$\psi = \theta_1(\phi_2) - \theta_2(\phi_2) \tag{20}$$

is independent of the concentration ϕ_2 and we obtain $\psi = \theta_{11} - \theta_{21} = \theta_{12} - \theta_{22}$. The angle ψ is determined by the shape and size of the species and therefore determines the degree of segregation. If species 1 are the smallest, then $\theta_1(\phi_2) > \theta_2(\phi_2)$ for any composition of the surface ϕ_2 (i.e., the small grains are always captured first and also $\theta_{12} > \theta_{21}$). The repose angles of the pure species, θ_{11} and θ_{22}, are intermediate between θ_{12} and θ_{21}. For a mixture of grains with different shape and friction coefficients $\theta_{11} \neq \theta_{22}$, and $\theta_{12} = \theta_{21}$ if the grains are of the same size.[1] If the species have the same shape and friction coefficients, then $\theta_{11} = \theta_{22}$.

Now, from Eqs. (18) and (7), we obtain

$$\Gamma = \Gamma_1 + \Gamma_2 = 2\gamma(\theta - \theta_1)R_1 + 2\gamma(\theta - \theta_2)R_2. \tag{21}$$

Substituting the value of Γ from Eq. (14) into Eq. (21), we get

$$2\gamma(\theta - \theta_1)R_1 + 2\gamma(\theta - \theta_2)R_2 = -\frac{vR^0}{L}. \tag{22}$$

From Eqs. (20) and (22), we get

$$\theta - \theta_1 = -\frac{\left(\frac{vR^0}{L} + 2\gamma\psi R_2\right)}{2\gamma R} \tag{23}$$

and

$$\theta - \theta_2 = -\frac{\left(\frac{vR^0}{L} - 2\gamma\psi R_1\right)}{2\gamma R}. \tag{24}$$

Combining Eqs. (5), (7), (9), (18), and (22), we obtain

$$\phi_1 = \frac{(C - \gamma(\theta - \theta_1))R_1}{\left(\frac{vR^0}{2L} + CR\right)}. \tag{25}$$

Substituting Eqs. (23) and (16) into Eq. (25), and transforming R_i into R_i^*, we further simplify:

$$\phi_1 = R_1^* + \frac{\gamma\psi}{C}\frac{R_1^*(1 - R_1^*)}{\left(1 + \frac{v}{2C}\frac{1}{(L - x)}\right)}. \tag{26}$$

Now, substituting Eq. (26) for the concentration into differential Eq. (17) and simplifying the differential, Eq. (17), we get

[1]For grains with different shape and same size the equality $\theta_{12} = \theta_{21}$ is not rigorously proven in the literature but we are not using this condition anywhere in our work.

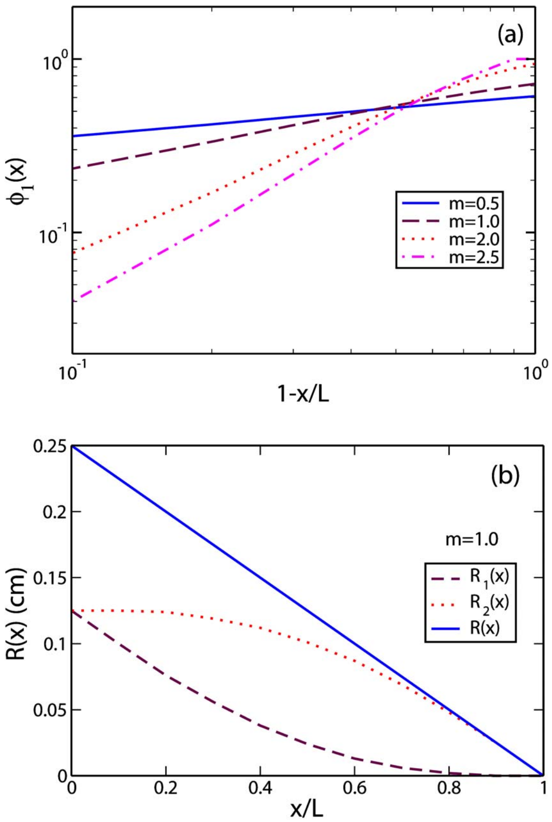

FIG. 5. (Color online) Steady-state solution for the two species granular flow in a two-dimensional silo in the case of partial segregation. (a) Concentration profile of small grains for different values of the power-law exponent m. (b) Profiles of the rolling phase for each species when $m=1.0$.

$$(L-x)\frac{\partial R_1^*}{\partial x} = -\frac{\gamma\psi}{C}\frac{R_1^*(1-R_1^*)}{\left(1+\frac{v}{2C}\frac{1}{(L-x)}\right)}. \tag{27}$$

Solving the differential Eq. (27), we obtain a power-law form for the rolling species:

$$R_1^*(x) = \frac{1}{\left[1+\frac{R_2^0}{R_1^0}\left(\frac{L+v/2C}{L-x+v/2C}\right)^m\right]}, \tag{28}$$

$$R_2^*(x) = \frac{\frac{R_2^0}{R_1^0}\left(\frac{L+v/2C}{L-x+v/2C}\right)^m}{\left[1+\frac{R_2^0}{R_1^0}\left(\frac{L+v/2C}{L-x+v/2C}\right)^m\right]}, \tag{29}$$

where the power-law exponent $m=\gamma\psi/C$ depends on the degree of difference between the species. The final expression for the concentration of small grains ϕ_1 is obtained when Eq. (28) is plugged into Eq. (26). We illustrate the steady-state concentration profile in Fig. 5(a) for different values of the power-law exponent m and for the case $R_1^0=R_2^0$. The profiles of the rolling species for $m=1$ are shown in Fig. 5(b). Also, we choose the following typical experimental values for the parameters in the calculations [20,30]: $L=30$ cm, $\gamma=20/\text{sec}$, $v=10$ cm/sec, $\psi=0.0-0.3$,[2] and $R^2=0.25$ cm.

The parameter C is set $C=1.2/\text{sec}$, and it is obtained by comparing our experimental data for $\phi_1(x)$ with the theoretical predictions of Eq. (26) [i.e., we plot the experimental data obtained for $\phi_1(x)$ as a function of $(1-x/L)$ and compare these data with Eq. (26), then perform a fitting procedure for the parameter C]. The procedure to obtain the free parameter C is the following: we first start with Eq. (26) and make an initial guess for the parameter C (considering that the analytical solution with that particular value of C should provide the maximum spectrum for the different degrees of mixing). Initially we choose a value of C equal to γ as our initial guess since both C and γ are collision rates. Using that initial guess in Eq. (26), we then perform a fit with the experimental data and then fine-tune the value of C with the experimental curve to find the best match. A value of $C=1.2/\text{sec}$ gives the best fit to the experimental results.

Figure 5(a) shows that particles 1 (small grains) have a higher concentration at the top and thus preferentially stay at the top of the pile while particles 2 (large grains) stay at the bottom. The degree of segregation is very small since the concentration varies very smoothly as a function of position. We observe that our model produces the same power-law behavior as the one proposed in the minimal model of BdG. It seems rather surprising that two models, starting from very different forms of collision functions, end up with the same kind of behavior. The answer to this question lies in the fact that the relevant parameter is the angular difference ψ between the generalized angle of repose and not the generalized angle of repose $\theta_i(\phi_j)$ [see Eqs. (19) and (20)]. In other words, our steady-state solution does not depend on the slope m_1 [Eq. (19)]. In fact, the constant m_1 does not appear anywhere in the equations for the steady-state solution. On the other hand, BdG in their work assumed the slope m_1 equal to zero. The profile of $R(x)$ [Fig. 5(b)] behaves linearly with x, which is the result of the conservation of the number of grains. Also, we observe slowly decreasing profiles for the rolling species $R_i(x)$.

We observe, however, that as we increase the power-law exponent $m(m>2)$, the power-law behavior does not seem suitable to model partial segregation [Fig. 5(a), $m=2.5$]. As discussed in Ref. [29] the angular difference ψ determines the degree of segregation. Thus complete segregation is expected for large values of ψ. When the size ratio is close to 1 [21,22], the angle ψ is expected to be small and we can linearize the collision functions [Eq. (18)]. When there is a large difference in size, we expect ψ to be large and the linear approximation for the collision functions breaks down. Since the exponent m is proportional to ψ (i.e., $m\propto\psi\propto d_2/d_1$), a large size ratio means large $m(m>2)$ and the power-law model is no longer applicable to calculate the

[2]Here we purposely choose $0\leqslant\psi\leqslant0.3$ [even though typical values for partial segregation are $0<\psi\leqslant0.1$ (Ref. [29])] in order to check whether the analytical solution for partial segregation is valid for higher values of ψ (i.e., for both partial and complete segregation).

steady-state solution. In this case, a complete-segregation effect acts in the system. In what follows, we will review the analytical solution for complete segregation obtained in Ref. [28], which will be used later in the experimental section, focusing on both partial- and complete-segregation studies.

D. Steady-state solution for complete segregation

We now focus on the case for which the difference in size or angle of repose is large, hence leading to complete or strong segregation. In this type of segregation, two distinct segregated regions for the two types of grains used are observed [see Fig. 1(b)]. This type of segregation is due to the collisions between the particles in the rolling and static phase, as captured by the BCRE formalism. However, in the case of thick flow there is an extra segregation mechanism helping the large grains to move to the bottom. When the grains are very different, the segregation takes place also in the flowing layer where small particles move in between the gaps of larger particles, leaving the latter on top of the rolling phase. This effect is known as percolation, kinematic sieving, or free surface segregation [20,29,30]. Thus, small rolling grains form a sublayer underneath the sublayer of large rolling grains. A thick rolling phase is a necessary condition for the percolation effect to take place. Therefore we may infer that there should not be any capture of larger particles in the upper region since the smaller particles screen the larger rolling grains with the surface. To simulate these effects, the following definitions of the collision functions were used in Ref. [28]:

$$a_i = x_i = \gamma\Pi[\theta(x) - \theta_i(\phi_i)],$$

$$b_i = \gamma\Pi[\theta_i(\phi_i) - \theta(x)], \tag{30}$$

where

$$\Pi[X] = \begin{cases} X & \text{if } X \geq 0, \\ 0 & \text{if } X < 0. \end{cases}$$

To calculate the steady-state solution of the equations of motion, calculations were done in two regions: the upper part of the pile (*region A*), where $\theta_2(\phi_2) < \theta < \theta_1(\phi_2)$, and the lower part of the pile (*region B*), where $\theta < \theta_2(\phi_2) < \theta_1(\phi_2)$. The steady-state solutions in these regions are the following.

Region A. In this region only smaller grains are present, defined in the region $0 \leq x < x_m$, where $x_m = R_1^0 L/R^0 - v/(\gamma\psi)$ defines the point where the pure region of smaller grains ends:

$$\phi_1(x) = 1, \tag{31a}$$

$$\phi_2(x) = 0 \tag{31b}$$

and

$$R_1(x) = R_1^0\left(1 - \frac{x}{L}\right) - R_2^0\frac{x}{L}, \tag{32a}$$

$$R_2(x) = R_2^0. \tag{32b}$$

Region B. Valid at the lower part of the pile ($x_m \leq x \leq L$), mainly larger grains are present after a small region of the order of $v/(\gamma\psi)$:

$$R_1(x) = \frac{R(x)}{\left[1 + A\exp\left(\frac{x - x_m}{r}\right)\right]}, \tag{33}$$

$$R_2(x) = \frac{R(x)A\exp\left(\frac{x - x_m}{r}\right)}{\left[1 + A\exp\left(\frac{x - x_m}{r}\right)\right]}, \tag{34}$$

where $A = R_2^0 L/R_1^0 r$ and the characteristic length of segregation is $r = v/\gamma\psi$. Also, the concentration profile as a function of the rolling species is given as

$$\phi_1(x) = \frac{R_1(x)}{R(x)}\left[1 + \frac{L}{rR^0}R_2(x)\right], \tag{35}$$

$$\phi_2(x) = \frac{R_2(x)}{R(x)}\left[1 - \frac{L}{rR^0}R_1(x)\right]. \tag{36}$$

The parameter r is expected to be of the order of the size of the grains, so for large system size $r \ll L$ and $A \gg 1$. The theoretical profiles of steady-state solutions for the following typical experimental values [20], and when $R_1^0 = R_2^0$, are shown in Fig. 6: $L = 30$ cm, $\gamma = 20/\text{sec}$, $v = 10$ cm/sec, $\psi = 0.1 - 0.3$, and $R^0 = 0.25$ cm.

Figure 6(b) shows complete segregation of the small grains at the top [$\phi_1(x=0) = 1, \phi_2(x=0) = 0$]. The concentration of the small grains then decays exponentially with a region of mixing of the order of the characteristic length r, formed at the center of the pile (in contrast to the power law formed in the case of partial segregation). We also notice that the characteristic length r or the parameter r/L decreases as the size ratio d_2/d_1 increases. Therefore, since $\psi \propto d_2/d_1$, increasing the size ratio makes the region of mixing small and the segregation becomes more and more complete. However, we observe that the model does not predict the correct behavior of the complete-segregation process when the parameter r/L is large [Fig. 6(b), $r/L = 0.20, 0.50$]. Figure 6(a) shows the same linearly decaying profile for $R(x)$ but exponential behavior is observed for the rolling species $R_i(x)$, compared to a slowly decaying profile in the case of partial segregation.

From our analytical study, we conclude that both the power-law and exponential models can predict the correct behavior of the segregation processes only within a limited regime. In the following section we will show the results from our experiments and predict the regime for applicability of both models in terms of the size ratio of the species and the respective model parameters.

III. EXPERIMENTAL STUDY

Now, we perform an experimental test of the prediction of the above theory by measuring the concentration profiles of

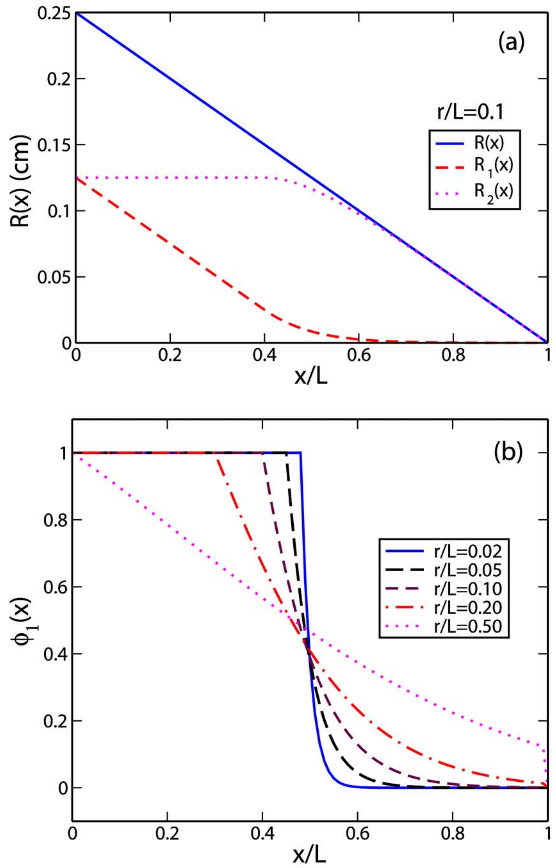

FIG. 6. (Color online) Steady-state solution for a granular flow of two species in the case of complete segregation. (a) Profiles of the rolling species when the parameter $r/L=0.1$. (b) Concentration profile of small grains for different values of the parameter r/L.

granular mixtures as a function of the size ratio between particles. Our experimental setup consists of a vertical quasi-two-dimensional silo with a very narrow gap separating two transparent glass plates. The length of the silo is $L=22$ cm in the lateral direction and the gap is $l=0.5$ cm. We close the three edges (left, right, and bottom) of the silo, and the granular mixture is poured from the right corner of the top.

In this study, we focus on both partial and complete types of segregation. To perform experiments with different size ratios of grains, d_2/d_1, various mixtures of grains composed of two species differing in size only are considered (we choose grains with no difference in shape so that the process of stratification can be avoided). We use acrylic beads of size 500–600 μm in diameter, spherical shape, as the first type of grains and acrylic or glass beads (spherical shape, different diameters for different experiments) as the second type of grains. We are not labeling the species here, since the labels 1,2 will be determined by the size ratio of the species. Since both the acrylic and glass beads are transparent in nature, to bring contrast in the image we color the 500–600 μm acrylic beads black using a dye. To perform experiments for different size ratios d_2/d_1, acrylic or glass beads of different sizes (90–130, 180–125, 850–1000, 360–425, 675–775, and 400–450 μm, respectively) are chosen. The information regarding the angles of repose of the pure species (θ_{ii}) is given in Fig. 7, where θ_{ii} is obtained by pouring the grains into a

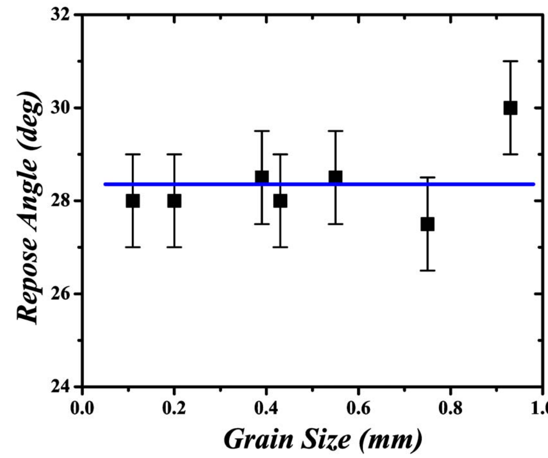

FIG. 7. (Color online) Angle of repose of acrylic beads of different size. All beads are quite spherical in shape.

silo and simply measuring the slope of the pile. We observe that the angle of repose of the pure species does not depend on the size of the particle if the species are of the same shape (in agreement with previous studies [20]). The experiments to measure the cross angle of repose involve the measurement of the kink as explained in Ref. [20]. In the present work, however, we deal only with segregation and not stratification, so we cannot measure the cross angle of repose as defined in the aforementioned work. Since the parameter ψ directly depends on the cross angles of repose ($\psi=\theta_{11}-\theta_{21}=\theta_{12}-\theta_{22}$), the experimental measurement of ψ from the values of cross angles is impossible. However, as demonstrated in published work [28,29], we consider ψ from 0.0 to 0.1 for partial segregation and from 0.1 to 0.3 for complete segregation. [3]

In all the experiments, an equal volume of two types of beads is poured at the top of the silo and we keep the flow rate constant. To capture the images, we use a Canon G6 digital camera. The digital images are then transferred to a workstation for image processing. The image analysis is done by analyzing the top few layers of the pile and by converting the color image into a gray-scale one. The variation in the gray-scale image is then calibrated to give the concentration of grains along the lateral direction. Figures 8(a) and 8(b) reflect two pictures of the experiments performed with two mixtures of different size ratios $d_2/d_1=1.29$ and $d_2/d_1=2.72$, respectively. Figure 8(a) shows partial segregation of a mixture of white glass beads ($d_1=400-450$ μm) and black acrylic beads

[3]The authors in Ref. [29] do not explicitly mention $0<\psi\leqslant 0.1$ in the case of partial segregation; however, it can be easily proven. The authors suggested that $\psi\propto d_2/d_1$ (i.e., proportional to the size ratio of the grains) and that ψ determines the degree of segregation (small values of ψ favor partial segregation and larger values of ψ favor complete segregation). They considered ψ of the order of 0.1–0.3 and $d_2/d_1\geqslant 1.4$ for complete segregation in their work. When $d_2/d_1<1.4$, a partial segregation of mixture is observed experimentally which clearly suggests that ψ must be of the order of 0–0.1 for partial segregation (since perfect mixing means ψ equals zero).

(a)

(b)

FIG. 8. (Color online) Experimental pictures of size segregation for different size ratio d_2/d_1 of black acrylic beads and white glass beads. (a) Example of partial segregation for a mixture of white glass beads (d_1=400–450 μm) and black acrylic beads (d_2=500–600 μm): average size ratio d_2/d_1=1.29. (b) Example of complete segregation for a mixture of white glass beads (d_1=180–225 μm) and black acrylic beads (d_2=500–600 μm): size ratio d_2/d_1=2.72.

(d_2=500–600 μm) with an average size ratio d_2/d_1=1.29. It is clear that the concentration of black beads (larger grains) is low at the top and it increases continuously as we go toward the bottom of the heap, whereas the concentration of the white beads (smaller grains) decreases as we reach the bottom. On the other hand, when a mixture of the same kind of beads but with a different size ratio d_2/d_1=2.72 (d_1=180–225 μm and d_2=500–600 μm) is poured into the silo, complete segregation of the mixture is observed. In this case the smaller white beads stay at the top, while the larger black beads stay at the bottom and the region of mixing is very small compared to partial segregation.

To compare our analytical and experimental results, we analyze the images and calculate the concentration profiles for the smaller grains along the lateral direction (Figs. 9 and 10). In both partial- and complete-segregation cases, we compare the two solutions for the following experimental values for the following parameters: L=22 cm, γ=20/sec,

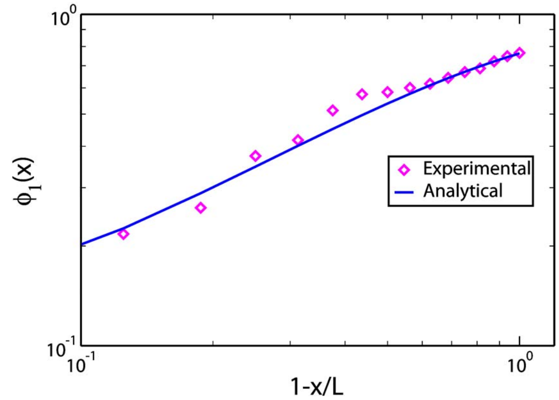

FIG. 9. (Color online) Comparison of the experimental concentration profile (of small grains) with steady-state analytical solutions for the power-law model. We pour a mixture of black acrylic beads and white glass beads with average size ratio d_2/d_1=1.29 into a two-dimensional silo. Good agreement is found between the experimental and analytical results, and we obtain a value equal to 1.25 for the power-law exponent m.

v=10 cm/sec, R^0=0.25 cm, R_1^0/R_2^0=0.5, and C=1.2/sec.

We have used the same experimental conditions as in previous studies by Ref. [20] and could reproduce the parameters for the velocity of the rolling grains v and γ. We used a high-speed video camera to track the motion of individual grains v. In our work we followed the same procedure and we could measure the same average value of the velocities of rolling grains. The other parameter γ=20/sec has been chosen by fitting the experiments and the analytical solution [20]. As mentioned earlier in the section on the analytical solution for partial segregation, the parameter C has been obtained by means of a fitting procedure.

Good agreement is found between the analytical and experimental results for both types of segregation, as seen in Figs. 9 and 10. For the case of partial segregation, both experimental data and analytical calculations produce a smooth power-law behavior. A value equal to 1.25 for the power-law

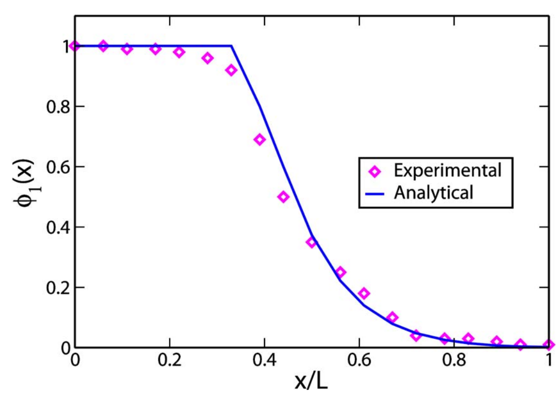

FIG. 10. (Color online) Comparison of the experimental concentration profile (of small grains) with steady-state analytical solutions for complete segregation. We pour a mixture of black acrylic beads and white glass beads with average size ratio d_2/d_1=2.72 into a two-dimensional silo. Good agreement is found between the experimental and analytical results, and the parameter r/L attains a value equal to 0.125.

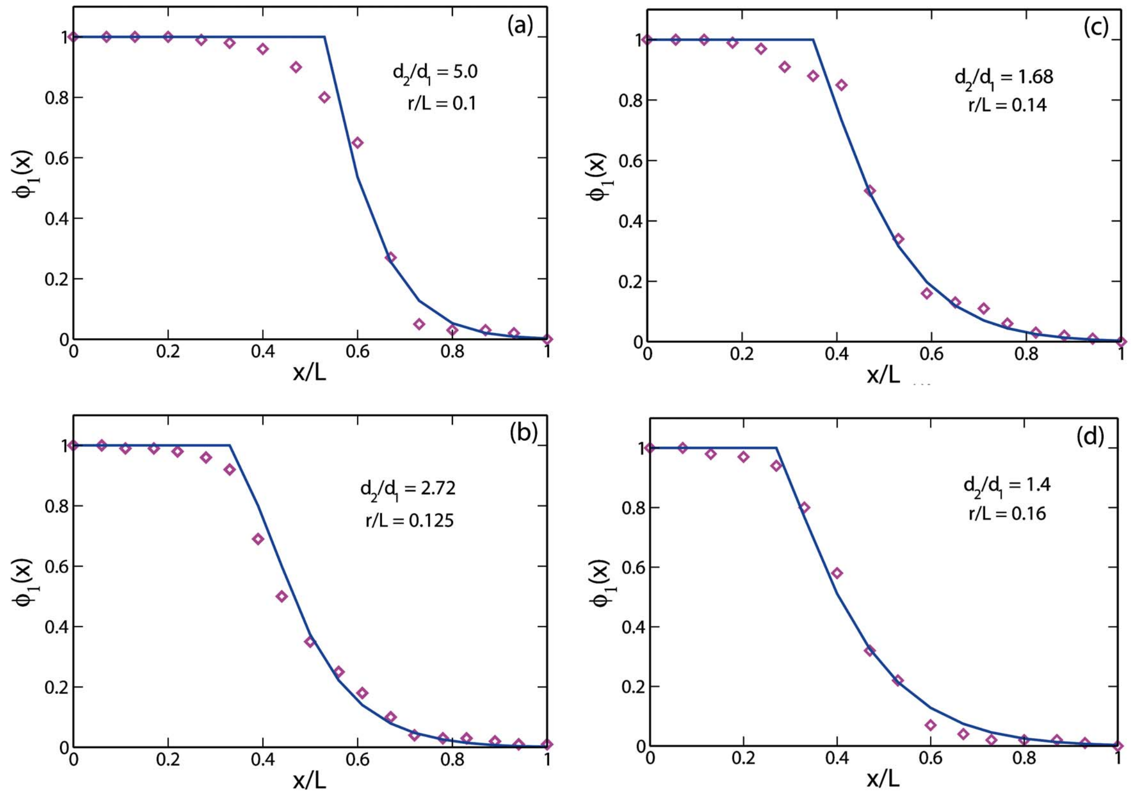

FIG. 11. (Color online) Comparison of the experimental concentration profile (of small grains) with the analytical solutions for complete segregation for different values of the size ratio d_2/d_1 of grains. The fitting parameter r/L is shown inside each plot. In all plots (a)–(d), the discrete points denote the experimental data and solid line denotes the analytical solution. We do not obtain a good agreement between experiments and the analytical model, when the size ratio d_2/d_1 is smaller than 1.4. In this case, the power-law model fits better (see Fig. 9).

component m is predicted by fitting the experimental data to the analytical solutions within the power-law approach (Fig. 9). In the case of complete segregation the experiment profile for the concentration of small grains decays exponentially with the position and a fit between the experimental data and the analytical solutions for complete segregation produces a value equal to 0.125 for the parameter r/L (Fig. 10).

To better understand the transition between the power-law and exponential segregation we perform experiments for different values of the size ratio d_2/d_1. In turn, we investigate the effect of the size ratio d_2/d_1 on the parameter r/L by conducting experiments with different mixtures of glass and acrylic beads. We perform similar comparisons of the experimental data with the analytical solutions for complete segregation profiles to obtain the fitting parameter r/L (Fig. 11). The plot of the parameter r/L with size ratio d_2/d_1 is shown in Fig. 12.

We observe that for large values of the size ratio d_2/d_1 ($d_2/d_1 \gtrsim 1.4$), the experimental data show good agreement with the analytical results for complete segregation [Figs. 11(a)–11(d)] and the fitting parameter r/L decreases slowly with size ratio d_2/d_1 (Fig. 12). For small values of the size ratio, $d_2/d_1 \lesssim 1.4$ (here $d_2/d_1 = 1.32, 1.29$), we do not obtain a good fit between the experiments and the theoretical model for complete segregation. In fact, a better fit is observed with the power-law approach (as seen earlier in Fig. 9 for $d_2/d_1 = 1.29$). These results clearly suggest that a complete-segregation model is applicable for large values of the size ratio $d_2/d_1 \gtrsim 1.4$ (and the model parameter $r/L \lesssim 0.16$, for our system). Below this critical value of the size ratio, the power-law model mimics the correct behavior of the segregation phenomenon.

We have calculated the parameter ψ from the determination of the exponent m and r. From the definition of the

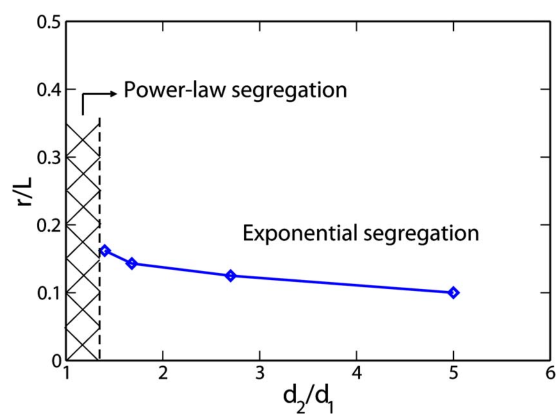

FIG. 12. (Color online) The fitting parameter r/L vs size ratio d_2/d_1 of the grains. We conduct experiments with different size ratio d_2/d_1 and obtain the parameter r/L by fitting the experimental data with the analytical solution for the exponential segregation. We find two regions based on the size ratio of the grains: power-law segregation region, if $d_2/d_1 \lesssim 1.4$, and exponential segregation region, if $d_2/d_1 \gtrsim 1.4$.

TABLE I. Calculation of ψ from the determination of the parameters m and r. The parameters m and r (or, equivalently, r/L) are obtained by fitting experimental data to the analytic solutions for partial and complete segregation, respectively.

d_2/d_1	Segregation type	m	$r\,(\text{cm}) = (r/L) \times L$	ψ
1.29	Partial	1.25		0.052
1.32	Partial	1.40		0.058
1.40	Complete		3.52	0.142
1.68	Complete		3.09	0.162
2.72	Complete		2.75	0.182
5.00	Complete		2.20	0.227

power-law exponent m, the parameter $\psi = mC/\gamma$ when segregation is partial and $\psi = v/(\gamma r)$ when the segregation is complete. Table I displays the calculated values of ψ for the different values of m and r for different size ratios (d_2/d_1) of the grains considered.

From Table I, we observe that the calculated values of ψ agree well with previous studies [28,29], where the authors suggest that the values for ψ are within the range ($0 < \psi \leqslant 0.1$ for partial segregation and $0.1 < \psi \leqslant 0.3$ for complete segregation).

IV. CONCLUSIONS

We find the analytical solutions for a model of surface flow of granular mixtures of two species, poured into a quasi-two-dimensional silo. Our model for partial segregation is based on the minimal model proposed by BdG, but we forego their approximations and consider more realistic collision functions to calculate the steady-state solution. The new collision functions capture the dependence of the angle of repose θ on the grain concentration ϕ. The proposed model predicts the same power-law dependence as found by BdG. Our results suggest that the relevant parameter in the calculations is the difference between the angle of repose (ψ) and not the angle of repose $\theta(\phi)$ itself. The case of exponential segregation is also discussed in the framework to compare the results of partial and complete segregation. Our experiments confirm the analytical profiles for both types of segregation and suggest the onset of a transition from complete to partial segregation at a critical size ratio $d_2/d_1 \simeq 1.4$ of the species, which has not been observed before. Below this critical value, partial segregation of the mixture is observed. We also predict the regime for applicability of both partial- and complete-segregation models in term of the size ratio of the species and the respective model parameters.

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